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REMARKS ON THE MACH EFFECT

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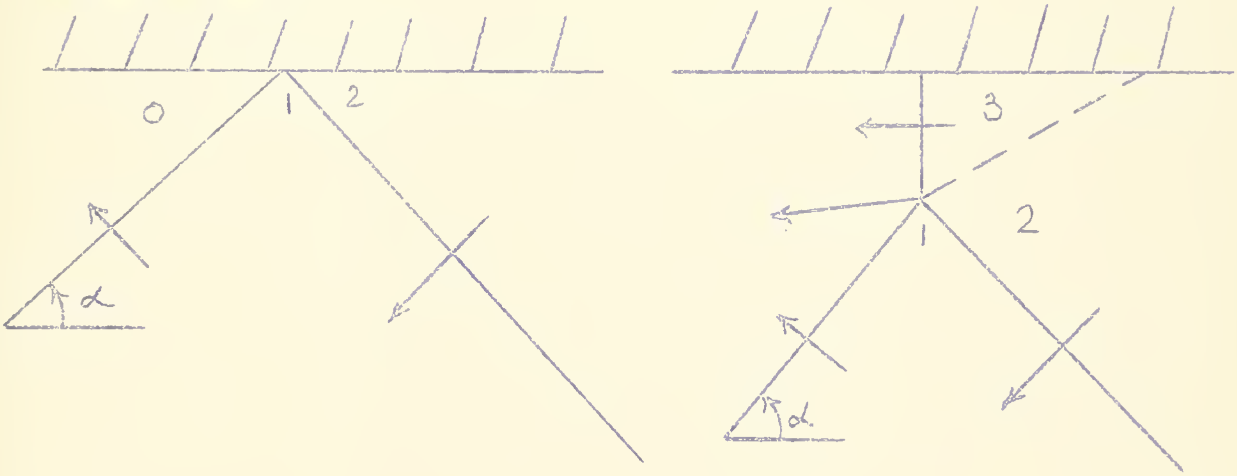
REMARKS ON THE MACH EFFECT

BY

K. O. Friedrichs.

Assume that a plane shock wave is moving toward a plane wall. If the angle of incidence α is small enough, a plane shock wave will be reflected. If, however, the angle of incidence is relatively large, a "Mach effect" will occur; the "branchpoint", i. e. the point at which the reflected wave will leave the incident wave, moves away from the wall and is connected to the wall by another plane shock wave, the "Mach shock".1) The aim of the present remarks is to discuss circumstances under which such a flow is possible and to determine the magnitude of the pressures which result behind the reflected wave.



Regular reflection. Mach effect.

Fig. 1

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1) Cf. v. Neumann, O. S. R. D Report 1140 p.18

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We denote the velocity of the by (u, v), the pressure by p, the density by ρ, the sound velocity by $c = (\gamma p \rho^{-1})^{1/2}$. The wall is at $y = 0$, the fluid in the half-plane $y < 0$. We characterize the states ahead of the incident wave, behind the incident, and behind the reflected wave, by (0), (1), and (2), respectively. We assume the fluid ahead of the incident wave to be at rest ($u_o = 0$) and its state (p_o, ρ_o) to be known. The incident wave is then characterized by the pressure ratio p_1/p_o and by the angle of incidence α (see Fig. 1). The situations to be investigated thus cover a strip

$$1 \leq p_1/p_o < \infty, \quad 0 \leq \alpha \leq \pi/2$$

in a $(p_1/p_o, \alpha)$-plane.

Regular reflection has been extensively investigated by von

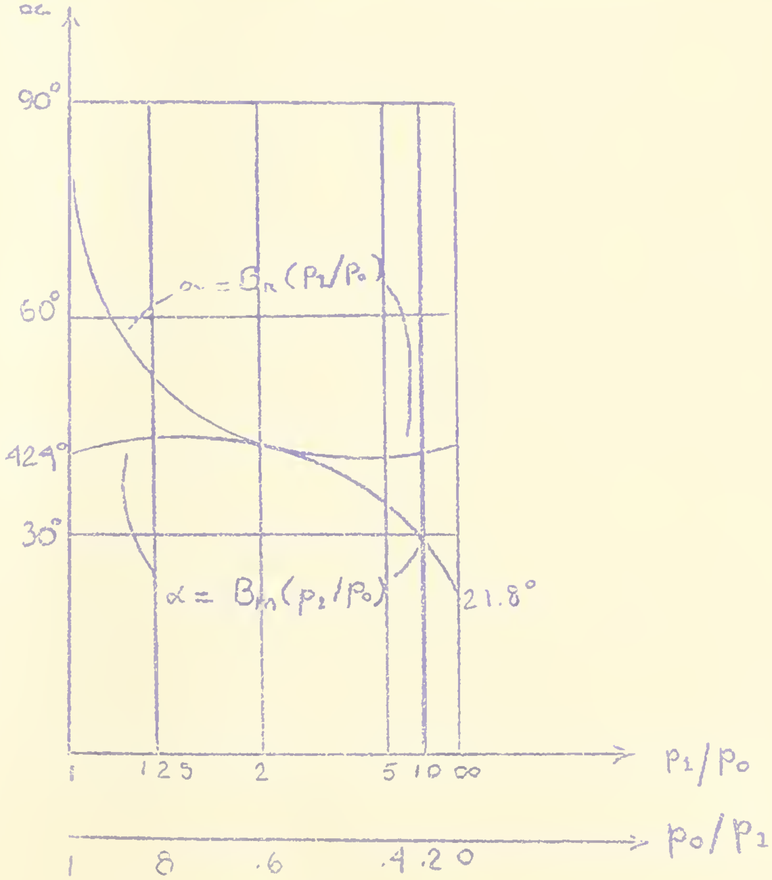

Fig 2

Neumann in collaboration with Seeger [2] Such a regular reflection is possible only if the angle α is not above a certain limit value depending on p_1/p_0 :

$$\alpha \leq B_r(p_1/p_0)$$

For $\alpha \leq B_r(p_1/p_0)$, two different reflections are possible, a "strong" and a "weak" one. It is assumed that in general the weak reflection will occur. The curve $\alpha = B_r(p_1/p_0)$ is indicated in Figure 2 (cf. v.Neumann).

The most important quantity to be discussed is the "reflected" pressure ratio p_2/p_1 . For $\alpha = 0$. this ratio coincides with that of head-on reflection,

$$p_2/p_1 = \frac{(\mu+2)p_1 - p_0}{\mu p_0 + p_1}, \quad \mu = \frac{\gamma+1}{\gamma-1} = 6 ;$$

it thus rises from 1 to $\mu + 2 = 8$ as p_1/p_0 rises from 1 to ∞. For fixed p_1/p_0 the ratio p_2/p_1 decreases somewhat as α increases; but, as v.Neumann and Seeger have found, it will eventually increase somewhat beyond the head-on value as α increases toward $B_r(p_1/p_0)$, for certain values of p_1/p_0.

The simplest configuration which could explain the Mach effect is a "three shock configuration" [3] It consists of the incident wave (01), the reflected wave (12), the "Mach

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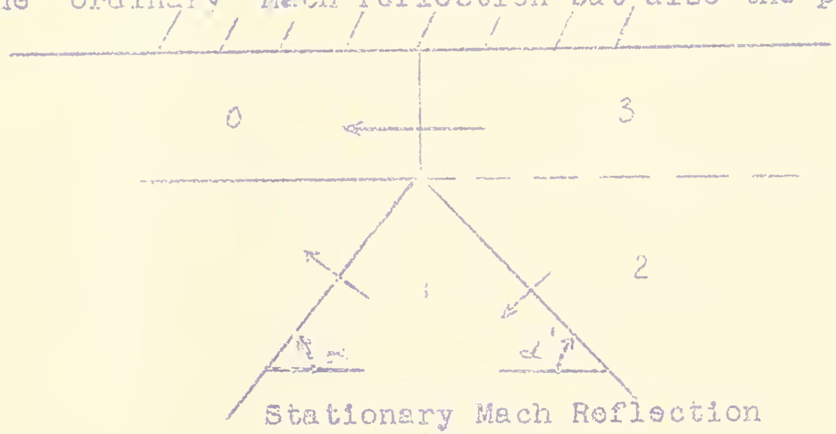
[2] In a report of which a draft is available. Cf. also a forthcoming A.M.P. Manual on Non-linear Wave Propagation.

[3] This term was introduced by Chandrasekhar who has investigated the possibility of three shock solutions in a different manner. Cf. Ballistic Research Lab Report 367 Aberdeen. As v.Neumann has pointed out orally experimental evidence seems to indicate that there are cases where the Mach effect cannot be described by a three shock configuration.

shock" (03), which is the perpendicular connection of the "branch point" with the wall. The regions (2) or (3) behind the reflected shock and the "Mach shock" are separated by a "separation line" (23), across which the pressure and the normal component of the flow velocity are continuous. The density may be different in the regions (2) and (3), and also the component of the flow velocity along the separation line. That is, the separation line is a vortex line. This follows simply from the fact that the fluid particles on both sides of the separation line have crossed different shocks. (It can be proved that no Mach effect without discontinuous tangential components at the separation line is possible; see A.M.P. Manual.)[4]

A three shock configuration is possible for every situation characterized by p_1/p_0 and α, provided one admits not only the "ordinary" Mach reflection but also the possibility



Stationary Mach Reflection
Fig. 3

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[4] At the wall the separation line will cause a reflected shock if the flow velocity in region (2) is supersonic when observed from a system moving with the velocity of region (3); otherwise a potential flow will result, destroying the assumed constancy of the flow in region (2).

that the branch-point moves toward the wall. Such "inverse" Mach reflection may occur, but it would eventually turn into a regular reflection. When only ordinary Mach reflection is considered, only a certain region of the $(p_1/p_0, \alpha)$-plane is covered. It is easily proved (see Appendix) that these situations will be limited by those in which the separation line is parallel to the wall. The corresponding configuration is a "stationary" Mach reflection, in which the branch point moves parallel to the walls and the Mach shock is a straight one.

These stationary three shock configurations can be determined analytically; the rather involved algebraic problem ultimately reduces to a quadratic equation. (See Appendix, section 3).

A set of such situations has been calculated for air, $\gamma = 1.4$; they lie on a curve[5]

$$\alpha = B_m(p_1/p_0)$$

in the basic rectangle; this curve connects the point $p_1/p_0 = 1$, $\alpha = 42.4^\circ$ with the point $p_1/p_0 = \infty$, $\alpha = 21.8^\circ$. Without calculations we conclude the basic fact that the B_m-curve lies below the B_r-curve

$$B_m(p_1/p_0) \leq B_r(p_1/p_0)$$

for every stationary Mach reflection can at the same time be interpreted as a regular reflection. As a matter of fact both

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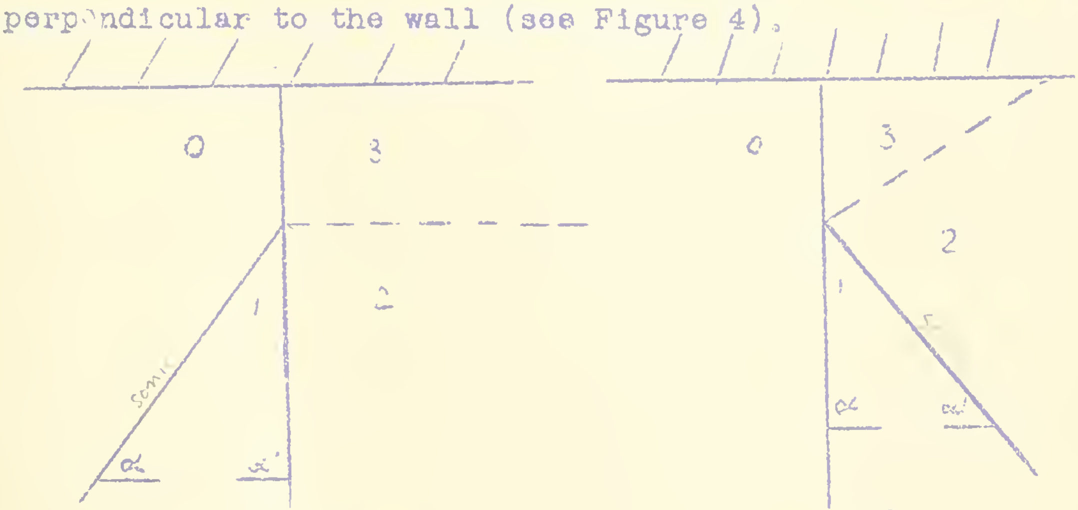
[5] The same curve has been determined in a different manner by Chandrasekhar and v. Neumann.

curves touch each other at about $p_1/p_0 = 2.31$. For $p_1/p_0 > 2.31$ the stationary Mach reflection is a weak one; for $p_1/p_0 <$ 2.31 ~~2.26~~, it is a strong one. A closer investigation indicates (cf. detailed discussion in the Appendix) for every state $\alpha > B_m(p_1/p_0)$ an ordinary three shock flow is possible. For $p_1/p_0 = 1$, $\alpha > 42.4^\circ$, the incident wave is sonic, while the reflected wave is perpendicular to the wall (see Figure 4).

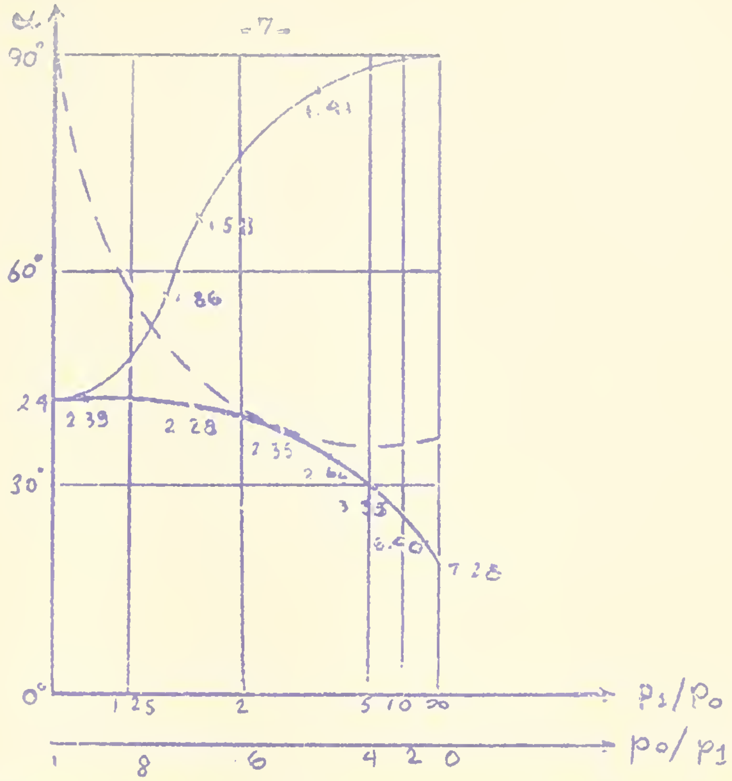


$p_1/p_0 = 1$ $\alpha = 90^\circ$

Fig. 4.

For $\alpha = 90^\circ$, the incident wave is perpendicular to the wall, the reflected wave is sonic. The vorticity of the separation line has shrunk to zero in both cases. These glancing three shock configurations can be determined explicitly (see Appendix, section 5).

The reflected pressure ratio p_2/p_1 has (cf. Fig. 5) the value 2.4 for $p_1/p_0 = 1$, $\alpha = 42.4^\circ$; it decreases to $p_2/p_1 = 1$ when α increases to 90°. Along the limiting



Reflected pressure ratio p_2/p_1 for the Mach effect

Fig. 5

curve $\alpha = B_m(p_1/p_0)$, the ratio p_2/p_1 first decreases slightly when p_1/p_0 increases, approaching the value $p_2/p_1 = 7.28$ for $p_1/p_0 \longrightarrow \infty$. The latter value is somewhat less than the value $p_2/p_1 = 8$ approached when $p_1/p_0 \longrightarrow \infty$ for the case of head-on collision, $\alpha = 0°$. For $p_1/p_0 = \infty$, the value p_2/p_1 decreases from 7.28 to 1 when α increases from 21.8° to 90°. For $\alpha = 90°$, we have $p_2/p_1 = 1$ throughout. It appears (cf. Appendix) that for every fixed incident pressure ratio p_1/p_0 the reflected ratio p_2/p_1 decreases to 1 from its value $\alpha = B_m(p_1/p_0)$ when α increases from this value to 90°.

Next consider what will happen when the angle of incidence α is increased from $\alpha = 0^{\circ}$ while the pressure ratio p_1/p_o is kept fixed. When α is slightly greater than zero, a regular reflection will occur. Let us assume that it is a weak one. When α has reached the value $B_m(p_1/p_o)$, the resulting flow in the regions (0), (1), and (2) will coincide with that for the limit Mach effect, provided $p_1/p_o > 2.31$. When α is further increased, probably the branchpoint will separate from the wall and a Mach flow will result, although a regular reflection is still possible as long as $\alpha \leq B_r(p_1/p_o)$. For $p_1/p_o < 2.31$, it seems more plausible that the regular reflection occurs up to $\alpha = B_r(p_1/p_o)$ and that then the configuration will jump over into that of the Mach flow.

The reflected pressure ratio p_2/p_1 for regular reflection behaves strangely when α increases while p_1/p_o is kept fixed (see v. Neumann). It first increases and when $\alpha \rightarrow B_r(p_1/p_o)$ increases again; if p_1/p_o is small it goes beyond the head-on value (i.e. for $\alpha = 0$). When the flow has turned over into the Mach flow, however, the reflected pressure ratio p_2/p_1 has a tendency to decrease; it will approach the value 1 when $\alpha \rightarrow 90^{\circ}$. (For about $p_1/p_o > 3$, stationary Mach reflection already occurs before p_2/p_1 for regular reflection starts rising

again.) In any case the reflected pressure ratio will never be much higher than for the head-on reflection, and in the fully developed three shock flow it will tend to become small. Generally we can say that the occurence of a three shock Mach effect results in a tendency to decreasing reflected pressure ratio, and the three shock Mach effect cannot serve to explain the occurrence of excessive pressures.

Tables

(Air, $\gamma = 1.4$)

In the following we give firstly a table for characterizing the stationary Mach configuration.[6] The velocity of the branch point is $(U, 0)$, the state (0) being at rest. The reference velocity is the "critical" sound velocity with respect to the branch point.

$$c_* = \sqrt{\frac{2}{\gamma+1}c_0^2 + \frac{\gamma-1}{\gamma+1}U^2} = 1.$$

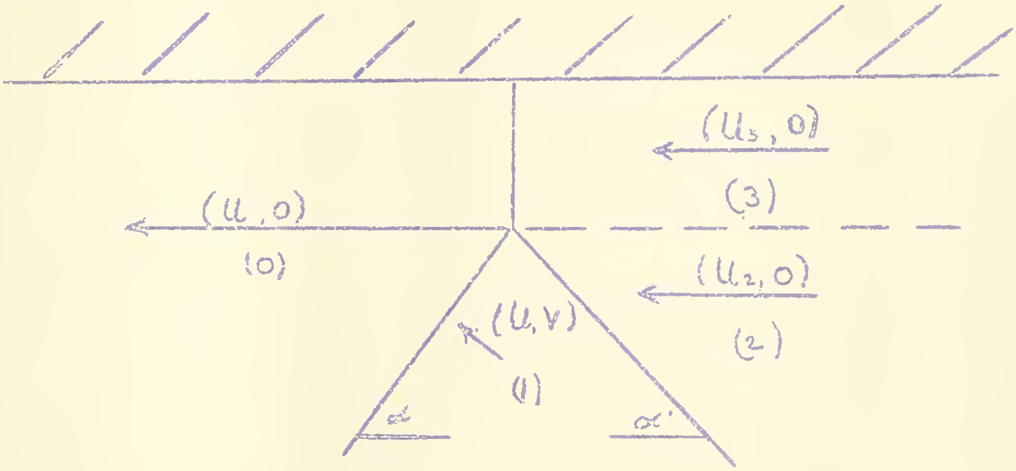


Fig. 6
Stationary three shock flow.

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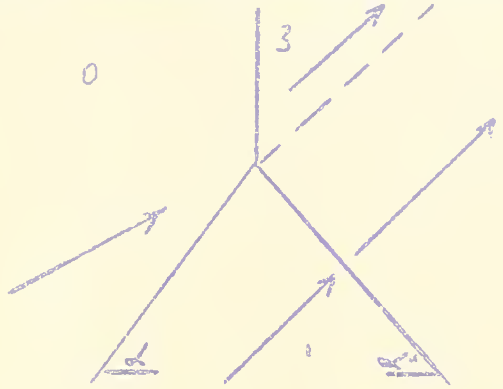
[6] From these values one reads off that

$$\frac{|u_2 - u_3|}{c_2} > 1 \qquad \text{for } p_1/p_0 > 5.48$$
$$< 1 \qquad \text{for } p_1/p_0 < 3.29$$

Consequently, near stationary Mach flow, the flow in region (2) will be supersonic when observed from (3) only for about $p_1/p_0 > 5$. Only for strong shocks therefore will the separation line be reflected at the wall as a shock, cf. footnote p.4.

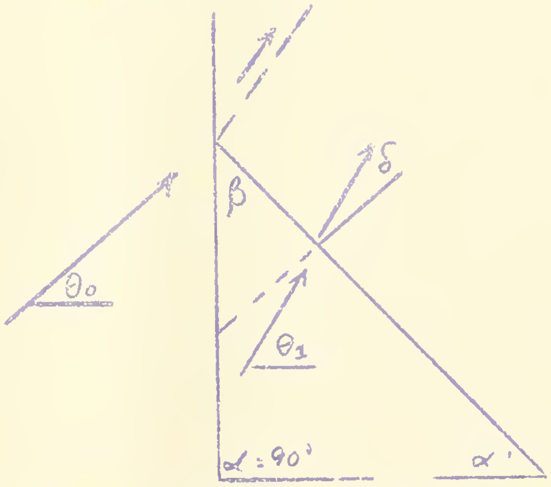
Secondly we give a few values characterizing certain ordinary Mach effects, viz. those for which the reflected shock is straight. These points are marked in Figure 5.



Three shock flow with straight reflected shock when observed from the branch point.

Fig. 7.

Thirdly we give a few values for the glancing Mach reflections, $\alpha = 90^0$.



Glancing three shock flow, $\alpha = 90^0$, observed from the branch point.

Fig. 8.

Table 1

,Stationary Three Shock Configuration

p_1/p_o	∞	50	5.48	3.29	2.140	1.491	1.067	1.000
p_2/p_1	7.28	6.40	3.36	2.66	2.34	2.28	2.39	2.400
p_2/p_D	∞	320	18.38	6.740	5.015	3.400	2.548	2.400
α in °	21.8	23.2	33.7	38.55	41.8	43.8	42.4	42.4
$\cos\alpha$	.929	.919	.832	.782	.745	.722	.738	.7384
$\sin\alpha$	.371	.394	.557	.623	.6675	.692	.675	.6743
α' in °	8.2	9.8	23.1	36.8	53.9	71.2	87.2	90.0
$\cos\alpha'$	.990	.986	.920	.810	.590	.324	.051	.000
$\sin\alpha'$	.144	.172	.392	.595	.808	.947	.999	1.000
ρ_1/ρ_0	6.000	5.4	2.95	2.23	1.70	1.33	1.05	1.00
ρ_2/ρ_1	4.04	3.2	2.26	1.96	1.81	1.77	1.83	1.833
ρ_2/ρ_0	24.26	17.1	6.66	4.37	3.07	2.355	1.92	1.833
ρ_3/ρ_0	6.00	5.9	4.37	3.63	2.83	2.275	1.91	1.833
a_0	∞	.148	.536	.690	.791	.863	.906	.913
a_1	.400	.447	.730	.837	.894	.915	.915	.913
a_2	.589	.636	.882	.978	1.019	1.037	1.044	1.045
a_3	1.080	1.078	1.075	1.070	1.063	1.055	1.046	1.045
$-U$	2.450	2.428	2.138	1.904	1.680	1.510	1.376	1.364
$-u_1$	.280	.308	.433	.408	.3085	.178	.028	0.000
v_1	.701	.718	.649	.512	.344	.186	.031	0.000
$-u_2$	.382	.433	.709	.787	.780	.721	.632	.616
$-u_3$	2.042	2.016	1.670	1.379	1.085	.847	.649	.616
p_1/p_o	∞	50.00	5.48	3.29	2.140	1.491	1.067	1.000
Head-on p_2/p_1	8.0	7.12	3.73	2.725	1.99	1.46	1.075	1.000

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Table 2

Three shock configuration with straight reflected shock

p_1/p_0	1.00	1.45	1.96	2.63	∞
p_2/p_1	2.40	1.88	1.58	1.41	1.00
α in °	42.4	56.0	65.8	80.9	90.0

Table 3.

Glancing three shock configuration

$\alpha = 90^\circ$

p_1/p_0	∞	20.00	7.02	2.00	1.33	1.11	1.05	1.00
p_0/p_1	0	.05	.1424	.50	.75	.90	.95	1.00
θ_0 in °	22.2	24.3	28.3	33.1	35.2	36.1	36.3	36.55
θ_1 in °	67.8	64.5	59.6	46.6	45.8	38.2	37.3	36.55
δ in °	0	3.75	9.5	24.7	31.45	34.7	35.65	36.55
$\beta = \theta_1 - \delta$	67.8	60.65	50.1	21.9	14.35	3.5	1.65	0
$\alpha' = 90^\circ - \beta$	22.2	29.35	39.9	68.1	85.65	86.5	88.35	90

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